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Review

Geometry of quantum spheres

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Abstract

Spectral triples on the q-deformed spheres of dimension two and three are reviewed.

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1. Introduction

The key notion of the most recent 'layer' of noncommutative differential geometry à la Connes [9] is *spectral triple*, which encodes the concept of a noncommutative Riemannian spin_c manifold.

In the classical (commutative) situation the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ canonically associated with a Riemannian spin_c manifold consists of the algebra $\mathcal{A}$ of smooth functions on M, of the Hilbert $\mathcal{H}$ space of (square integrable) Dirac spinors, carrying a representation of $\mathcal{A}$ (by pointwise multiplication), and of the Dirac operator D, constructed from the Levi–Civita' connection (metric preserving and torsion-free) plus a $U(1)$-connection.

When M is even dimensional there exists the grading (or parity) operator γ and when M is spin one also has operator J of real structure (known also as charge conjugation). All these data are of great importance both in Mathematics and Physics. They satisfy certain further seven properties which allow one to reconstruct back the underlying differential, metric and spin structures.

A generalization of these concepts to noncommutative algebras in the framework of noncommutative spectral geometry has already found plentiful applications. But the whole *zoo* of q-deformed spaces coming from the quantum group theory, was commonly believed not to match well the Connes' approach. This was supported by some apparent "no-go" hints such as that exponentially growing spectrum of the quantum Casimir operator would prevent bounded commutators with the algebra, some known differential calculi seemed not to come as bounded commutators with any D, an early classification of equivariant representations missed the spinorial ones and also on some deformation theory grounds. However, the intense recent activity indicated a possibility to reconcile these two lines of mathematical research.

In this paper spectral triples on some of the simplest studied examples of quantum spheres of lowest dimension (2 and 3) are reviewed. More precisely, we shall be concerned mainly with (the algebra of) the underlying space of the quantum group $SU_q(2)$ and its two homogeneous spaces known as the standard and the equatorial Podleś sphere (see [15] for the list of other low dimensional spheres).

For the sake to be consistent as far as possible with the conventions and notation for different examples we often give the original formulae in an equivalent form. In the sequel $0 < q < 1$ and $[x] = [x]_q$, where $[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$ for any number x.

2. Noncommutative Riemannian spin manifolds

2.1. Spectral triples and their equivariance

We recall the general definition.

Definition 1. A (compact) *spectral triple* $(A, \mathcal{H}, D)$ consists of a (unital) $*$-algebra A of bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D = D^\dagger$ on $\mathcal{H}$ with

(i) compact resolvent $(D-\lambda)^{-1}$, $\forall\lambda \in \mathbb{C} \setminus \mathrm{spec} D$,
(ii) bounded commutators $[D, \alpha]$, $\forall\alpha \in A$.

A spectral triple is called *even* if a grading γ of $\mathcal{H}$ is given, $\gamma^2 = 1$, such that $\alpha \in A$ are even operators, $\alpha\gamma = \gamma\alpha$, and D is odd $D\gamma = -\gamma D$. Otherwise it is called *odd* (by convention then $\gamma = 1$).

A spectral triple is called *real* if an antilinear isometry J is given, whose adjoint action sends A to its commutant

$$[\alpha, J\beta J^{-1}] = 0, \quad \forall\alpha, \beta \in A. \tag{1}$$

We are particularly interested in spectral triples on homogeneous embeddable quantum spaces, described by a comodule subalgebra A of a Hopf $*$-algebra H. It is natural to employ their symmetry in order to reduce the search freedom or to find *equivariant triples*. The formulation of the symmetry in terms of *coaction* and *coproduct* in H has not yet been presented but one can work with the dual quantum group.

To deal with the equivariance we shall use *action* of a Hopf $*$-algebra $\tilde{U}$. Typically $\tilde{U}$ is dual of H in the sense of nondegenerate Hopf algebra pairing (e.g. a quantum universal enveloping algebra). This yields two commuting (left and right) $\tilde{U}$-module algebra structures on H

$$u \triangleright h = h_1(u, h_2), \quad h \triangleleft u = (u, h_1)h_2, \tag{2}$$

where we use Sweedler's type notation for the coproduct in H. The star structure is compatible with both actions:

$$u \triangleright h^* = ((Su)^* \triangleright h)^*, \quad h^* \triangleleft u = (h \triangleleft (Su)^*)^*, \quad \forall u \in U, h \in H, \tag{3}$$

where S is the antipode in $\tilde{U}$.

The actions (2) can be combined into a (left) action of $\tilde{U} \otimes \tilde{U}^{op}$ via $(u \otimes u')\triangleright x := u\triangleright x\triangleleft u'$. (Another option is to pass to the left action of $\tilde{U} \otimes \tilde{U}$ by using S and a suitable automorphism anti co-homomorphism of U.)

In case A is a proper subalgebra of H, we assume that (at least) some nontrivial Hopf subalgebra U of $\tilde{U} \otimes \tilde{U}^{op}$ survives the restriction to A.

We shall use the following general definition.

Definition 2. Let U be a Hopf $*$-algebra and A be a U-module $*$-algebra. A spectral triple $(A, \mathcal{H}, D)$ is called *U-equivariant* if there exists a dense subspace V in $\mathcal{H}$ such that

(i) V is a module over $A \rtimes U$, so in particular

$$u(\alpha v) = (u_1 \triangleright \alpha)(u_2\, v), \quad \forall u \in U, \alpha \in A, v \in V; \tag{4}$$

(ii) $u^* \subset u^\dagger$ on V (as unbounded operators);
(iii) $D(V) \subset V$ and D is invariant

$$Du = uD \text{ (on } V), \quad \forall u \in U. \tag{5}$$

Moreover, for U-equivariant even and/or real spectral triples we shall require that $\gamma(V) \subset V$, $u\gamma = \gamma u$ and/or that J is the phase part of some $\tilde{J}$, which satisfies $\tilde{J}(V) \subset V$, $u\tilde{J} = \tilde{J}(Su)^*$, $\forall u \in U$.

(In (4) Sweedler's type notation is used for the coproduct in U and $\triangleright$ for the action of U on A.)

2.2. Further requirements

On top of the above a series of further seven requirements (axioms) [10] (see also [20]) are required to describe a noncommutative (compact Riemannian spin) manifold, that we briefly recall.

A spectral triple (A, H, D) is *regular* (or *smooth*) iff

$$[|D|, \dots [|D|, \beta \underbrace{] \dots]}_{n}$$

are bounded $\forall\, n \in \mathbb{N}$, $\beta \in \mathcal{A} \cup [D, \mathcal{A}]$.

This condition permits to introduce the analogue of Sobolev spaces $\mathcal{H}^s := Dom(|D|^s)$ for $s \geq 0$ (assume that $\mathcal{H}^\infty := \cap_{s\geq 0} H^s$ is a core of $|D|$). Then $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ has *analytic order k* iff T extends to a bounded $T : \mathcal{H}^{k+s} \to \mathcal{H}^s$, $\forall s \geq 0$.

It turns out that $\mathcal{A}(\mathcal{H}^\infty) \subset \mathcal{H}^\infty$. Moreover [14] (see also [21]) certain algebra $\mathcal{D} = \cup \mathcal{D}_k$ of *differential operators* can be introduced as the smallest algebra of operators on $\mathcal{H}^\infty$ containing $A \cup [D, A]$ and filtered by the analytic order $k \in \mathbb{N}$ in such a way that $[D^2, \mathcal{D}_k] \subset \mathcal{D}_{k+1}$. Next the space Ψ_k of *pseudodifferential operators* of order $k \in \mathbb{Z}$ consists of those $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ which for any $m \in \mathbb{Z}$ (especially $m \ll 0$) can be written in the form

$$T = S_0\,|D|^k + S_1\,|D|^{k-1} + \cdots + S_{k-m}\,|D|^m + R,$$

where $S_j \in \mathcal{D}_j$ and R has analytic order $\leq m$. (Here it is assumed that $|D|$ is invertible, otherwise one works e.g. with $\sqrt{1 + D^2}$.)

It can be seen that Ψ_0 is the algebra generated by the elements of $A \cup [D, A]$ and their iterated commutators with $|D|$, and that $[D^2, \Psi_k] \subset \Psi_{k+1}$. The algebra structure on $\Psi := \cup_{k\in\mathbb{Z}} \Psi_k$ can be read in terms of *asymptotic expansion*: $T \approx \sum_{j\in\mathbb{N}} T_j$, whenever T and T_j, for $j \in \mathbb{N}$, are operators $\mathcal{H}^\infty \to \mathcal{H}^\infty$ and $\forall m \in \mathbb{Z}$, $\exists\, N$ such that $\forall M > N$, $T - \sum_{j=1}^{M} T_j$ has analytic order $\leq m$. For instance for complex powers of $|D|$ (defined by the Cauchy formula) one has for $T \in \Psi$

$$[|D|^{2z}, T] \approx \sum_{j\geq 1} \binom{z}{j} [D^2, \dots [D^2, T \underbrace{] \dots]}_{j} |D|^{2z-2j}.$$

The algebra Ψ provides a convenient framework to study the residues of the zeta functions. For that it suffices that *dimension* requirement holds: $\exists\ n \in \mathbb{N}$ s.t. the eigenvalues (with multiplicity) of $|D|^{-n}$, $\mu_k = O(k^{-1})$ as $k \to \infty$. (The coefficient of the logarithmic divergence of $\sigma_N := \sum^{N} \mu_k$, denoted $⨍ |D|^{-n}$, defines the 'noncommutative integral'.) Then it follows that for $k > n$, D^{-k} is trace-class, i.e. *finitely summable*.

Let Σ be the *dimension spectrum*, i.e. the set of the singularities of zeta functions

$$\zeta_\beta(z) = \mathrm{Trace}_{\mathcal{H}}(\beta\,|D|^{-z}) \tag{6}$$

for any $\beta \in \Psi_0$. If Σ is assumed to be discrete with poles only as singularities then $\forall\, T \in \Psi_k$ the zeta function $\mathrm{Trace}(T|D|^{-z})$ is holomorphic in a half plane of $\mathbb{C}$ with $\Re z$ sufficiently large and has a meromorphic continuation to $\mathbb{C}$ with all poles contained in $\Sigma + k$. If in addition Σ contains only simple poles then the residue functional

$$\tau(T) := Res_{z=0}\mathrm{Trace}(T|D|^{-z})$$

is tracial on $T \in \Psi$ (c.f. [30]).

These tools serve for the local index theorem of Connes–Moscovici [14], which provides a powerful algorithm for performing complicated local computations by neglecting plethora of irrelevant details.

Another analytic requirement is *finiteness and absolute continuity* which states that A admits a (Fréchet) completion $\mathcal{A}$ which is a pre C^*-algebra and $\mathcal{H}^\infty$ is finite generated projective left $\mathcal{A}$-module.

Among the algebraic conditions for a spectral triple we shall use the *reality condition* which requires that $(A, \mathcal{H}, D)$ is real (in the sense stated above) and that certain sign conditions are satisfied for the square of J and its (anti) commutation with γ and with D, and the *first order condition* which requires that

$$[[D, \alpha], J\beta J^{-1}] = 0, \quad \forall \alpha, \beta \in A, \tag{7}$$

is satisfied. (The absence of these two properties should be interpreted as dealing rather with a spin$_c$ manifold and Finsler metric).

There are two more algebraic conditions. The first is *orientability*: there exists a Hochschild n-cycle

$$\sum c_0 \otimes c_0' \otimes c_1 \otimes \cdots \otimes c_n \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op}), \tag{8}$$

such that

$$\sum c_0 J(c_0')^* J^{-1}[D, c_1] \cdots [D, c_n] = \gamma, \tag{9}$$

where n is the dimension and γ the gradation operator. The second one is *Poincaré duality*, which can be formulated as the requirement that the pairing

$$K_0(\mathcal{A}) \times K_0(\mathcal{A}) \mapsto \mathbb{Z}, ([p], [q]) \mapsto \tfrac{1}{2}\,\mathrm{Index}(pJqJ^{-1}(1 + \gamma D\gamma)pJqJ^{-1}) \tag{10}$$

and

$$K_1(\mathcal{A}) \times K_1(\mathcal{A}) \mapsto \mathbb{Z}, ([u], [v]) \mapsto \frac{1}{4}\,\mathrm{Index}\left(\left(1 + \frac{D}{|D|}\right) uJvJ^{-1}\left(1 + \frac{D}{|D|}\right)\right) \tag{11}$$

is nondegenerate.

2.3. Reconstruction theorem

As stressed by Connes, the role played by D is two-fold: as the K-homology fundamental class (i.e. the *index* aspect) but also as the inverse of the infinitesimal unit of length *ds* (i.e. *ds* identifies with the *bare* Dirac propagator).

It is important to note that a spectral triple satisfying the above seven axioms encodes all the essential geometric information, which can be reconstructed back from it. This is certainly the case for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ under the assumption that $\mathcal{A} = C^\infty(M)$, where M is a smooth compact manifold. Its dimension n can be read off form the dimension axiom (which is nothing but a reformulation of the Weyl formula for the growth of eigenvalues of D). Next the metric and spin structure can be extracted by the *Reconstruction* Theorem of Connes [10]:

(a) There exists a unique Riemannian metric g on M such that the geodesic distance between any two points $x, y \in M$ is given by

$$d_D(x, y) = \sup\{|a(x) - a(y)| : a \in A, \|[D, a]\| \leq 1\}. \tag{12}$$

(b) g only depends upon the unitary equivalence classes, which form a finite collection of affine spaces Ω_σ parametrized by the spin structures σ on M.

(c) The action functional

$$Res_W(D^{2-n}) := \frac{1}{n(2\pi)^n} \int_{S^*M} tr(\sigma_{-n}(x, \xi))\, \mathrm{d}x\, \mathrm{d}\xi, \tag{13}$$

where $\sigma_{-n}(x, \xi)$ is order $-n$ part of the total symbol of D^{2-n} and *tr* is a normalized Clifford trace, is a positive quadratic form on each Ω_σ with a unique minimum given by the canonical classical spectral triple, at which (up to a constant) it attains the value equal to the Einstein Hilbert action of g, i.e.

$$\int R\sqrt{g}\, d^n x, \tag{14}$$

where R is the scalar curvature. We refer to [20] for the proof of the theorem.

Remarks. Though (12) has a 'dual' form to the usual $d(x, y)$ given by *infimum* over the geodesic lengths, they are in fact equal (by integrating a derivative of a function along a path one gets that $d_D(x, y) \leq d(x, y)$; the inequality being saturated for the function which just measures the length along the path).

The expression (13) given by Wodzicki residue (which on the appropriate class of operators agrees with the noncommutative integral) is well defined because by combination of axioms it can be shown that D is a pseudodifferential operator on M.

The formula (14) follows by realizing that $Res_W(D^{2-n})$ is (proportional) to the integral of the second coefficient of the heat kernel expansion of D^2 (and does not depend upon couplings to A_μ in case of spin$_c$ structure).

Connes conjectured that actually for any commutative $\mathcal{A}$ it can be deduced from the axioms (notably the Poincare' duality) that the spectrum X of $\mathcal{A}$ is a smooth manifold and that the map $X \to \mathbb{R}^N$, given by the finite collection $a^i_j \in \mathcal{A}$ involved in the Hochschild cycle c of orientation axiom, is actually a smooth embedding of M as a submanifold of $\mathbb{R}^N$. For the reconstruction of the manifold structure of X, see [27].

2.4. Connections with General Relativity

It is worth to mention that in [4,5] a 'universal' spectral action formula has been proposed to govern the dynamics including General Relativity

$$Tr\,\chi\left(\frac{D}{\Lambda}\right) + \text{matter}, \tag{15}$$

where χ is the (smoothed) characteristic function of [0,1] and Λ is a 'scale' parameter. In dimension 4 the first term approximates the Einstein–Hilbert action with a large cosmological term for "slowly varying" metrics with small curvature. Indeed, by the heat kernel expansion

$$S_G(D) = \Lambda^4 f_0 \int_M \sqrt{g}\,\mathrm{d}x + \Lambda^2 f_2 \int_M R\sqrt{g}\,\mathrm{d}x + \cdots,$$

where f_0 and f_2 are certain moments of χ.

The phase space Γ of General Relativity can be taken as Ricci flat metrics (solutions of the Einstein equations) modulo the (connected component) of the diffeomorphism group $\mathrm{Diff}_0(M)$. The observables are just real functions on Γ.

On compact M there is an infinite discrete family of the (ordered) real eigenvalues λ_n of D which are invariant under $\mathrm{Diff}_0(M)$. Assuming that (at least locally) they can be viewed as functions on Γ (i.e. depending on g) in [23] (by quantum mechanical perturbation theory and integration by parts) the Jacobian matrix of $g_{\mu\nu}(x) \to \lambda_n$ was computed and their Poisson brackets expressed in terms of the energy-momentum tensors and of the propagator of the linearized Einstein equations.

The action (15) can be expressed in terms of λ_n's but to get the equations of motion one cannot vary with respect to the λ_n's, which are not independent variables but satisfy the (unknown) constraints in order to arise from D of some geometry. (Variation with respect to the metric field produces the equations which are solved by saying that above Λ the energy momentum tensor of the nth eigenspinor scales as λ_n [23].)

It is possible that the problem of unknown constraints for λ_n (and of the transcendental dependence on D) can be overcome in the approach of Connes, i.e. by taking the spectral triples themselves as the right variables for General Relativity. Though the constraints are highly nonlinear e.g. the orientation axiom is n-linear in D in dimension n, at least they can be spelled out.

Despite some limitations (compactness of M and positivity of g) this should be worth to pursue e.g. in view of exceptionally simple (linear in D) universal action functional for spectral triples.

In the noncommutative realm the spectral triples and spectral geometry a la' Connes (a generalization of Riemannian spin manifolds) has already found several applications. But, as mentioned in Section 1, it was commonly believed that the q-deformed spaces coming from the quantum group theory do not to match this approach (as supported by some apparent 'no-go' hints). It turns out however that these two areas of mathematical research can be reconciled as I will illustrate on the examples of quantum spheres of lowest dimension (2 and 3).

3. Quantum three-sphere

In this section we review the spectral triples on the quantum three-sphere underlying the quantum group $SU_q(2)$. We start by recalling its definition, the quantum symmetry algebra and its representations.

The unital $*$-algebra $A(SU_q(2))$ [31] $(0 < q < 1)$ is generated by α, β satisfying

$$\alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha, \quad \beta\beta^* = \beta^*\beta, \quad \alpha^*\alpha + \beta^*\beta = 1, \quad \alpha\alpha^* + q^2\beta\beta^* = 1\,.$$

The classical subset is (the 'equator') S^1 given by the characters $\beta \mapsto 0, \alpha \mapsto \lambda$ with $|\lambda| = 1$. The C^*-algebra $\mathcal{A}$ associated to A is isomorphic to the extension of $C(S^1)$ by $\mathcal{K} \otimes C(S^1)$. The K-groups are $K_0 = \mathbb{Z}$, $K_1 = \mathbb{Z}$.

The dual (infinitesimal) symmetry of A can be described using the quantized algebra $U_q(su(2))$, which is a $*$-Hopf algebra with generators e, f, k, k^{-1} satisfying relations

$$ek = qke, \quad kf = qfk, \quad k^2 - k^{-2} = (q - q^{-1})(fe - ef),$$

and the coproduct

$$\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f.$$

Its counit ϵ, antipode S, and $*$-structure are given respectively by

$$\begin{aligned} &\epsilon(k) = 1, \quad \epsilon(e) = 0, \quad \epsilon(f) = 0, \\ &Sk = k^{-1}, \; Sf = -qf, \; Se = -q^{-1}e, \\ &k^* = k, \qquad e^* = f, \qquad f^* = e. \end{aligned}$$

The (commuting) left and right actions (2) of $U_q(su(2))$ on the generators of $A(SU_q(2))$ read explicitly

$$\begin{aligned} &k \triangleright \alpha = q^{-1/2}\alpha, \; k \triangleright \alpha^* = q^{1/2}\alpha^*, \; k \triangleright \beta = q^{1/2}\beta, \; k \triangleright \beta^* = q^{-1/2}\beta^*, \\ &f \triangleright \alpha = 0, \qquad f \triangleright \alpha^* = -\beta^*, \quad f \triangleright \beta = q^{-1}a, \; f \triangleright \beta^* = 0, \\ &e \triangleright \alpha = q\beta, \qquad e \triangleright \alpha^* = 0, \qquad e \triangleright \beta = 0, \qquad e \triangleright \beta^* = -\alpha^*, \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\alpha \triangleleft k = q^{-1/2}\alpha, \; \alpha^* \triangleleft k = q^{1/2}\alpha^*, \; \beta \triangleleft k = q^{-1/2}\beta, \; \beta^* \triangleleft k = q^{1/2}\beta^*, \\ &\alpha \triangleleft f = -b^*, \quad \alpha^* \triangleleft f = 0, \qquad \beta \triangleleft f = q^{-1}\alpha^*, \; \beta^* \triangleleft f = 0, \\ &\alpha \triangleleft e = 0, \qquad \alpha^* \triangleleft e = q\beta, \qquad \beta \triangleleft e = 0, \qquad \beta^* \triangleleft e = -\alpha. \end{aligned} \tag{17}$$

They can be combined into one (left) action of $U_q(su(2)) \otimes U_q(su(2))$ by

$$(u \otimes u')\triangleright x := u\triangleright x \triangleleft (S^{-1}\Theta(u')),$$

where $\Theta : k \mapsto k^{-1}, e \mapsto -f, f \mapsto -e$.

We recall the irreducible finite dimensional representations σ_ℓ of $U_q(su(2))$ labeled by spin $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$$\begin{aligned} f\,\epsilon_{\ell,m} &= [\ell - m]^{1/2}[\ell + m + 1]^{1/2}\,\epsilon_{\ell,m+1},\\ e\,\epsilon_{\ell,m} &= [\ell - m + 1]^{1/2}[\ell + m]^{1/2}\,\epsilon_{\ell,m-1}, \quad k\,\epsilon_{\ell,m} = q^m\,\epsilon_{\ell,m}, \end{aligned} \tag{18}$$

where the vectors $\epsilon_{\ell m}$, for $m = -\ell, -\ell+1, \ldots, \ell - 1, \ell$, form a basis for the irreducible U-module V_ℓ. In fact σ_ℓ are $*$-representations of $U_q(su(2))$, with respect to the hermitian product on V_ℓ for which the vectors $\epsilon_{\ell m}$ are orthonormal.

We return to the quantum three-sphere $SU_q(2)$.

Let χ be the normalized Haar state on $\mathcal{A}$, and let $L^2(SU_q(2))$ be the Hilbert space associated with χ. Take the orthonormal basis of $L^2(SU_q(2))$ as

$$\varepsilon_{\ell,i,j} := [2\ell + 1]^{1/2} q^{-i} t^\ell_{i,j}, \quad \ell \in \tfrac{1}{2}\,\mathbb{N}, i, j \in \{-\ell, -\ell + 1, \ldots, \ell\}, \tag{19}$$

where t^ℓ_{ij} is (i, j)th matrix element of the unitary irreducible corepresentation of spin ℓ.

We shall use the left regular unitary representation of $\mathcal{A}$ in $L^2(SU_q(2))$ which on the generators reads explicitly

$$\begin{aligned} \alpha\,\varepsilon_{\ell,i,j} = {} & q^{2\ell+i+j+1}\frac{(1 - q^{2\ell-2j+2})^{1/2}(1 - q^{2\ell-2i+2})^{1/2}}{(1 - q^{4\ell+2})^{1/2}(1 - q^{4\ell+4})^{1/2}}\,\varepsilon_{\ell+1/2,i-1/2,j-1/2}\\ & + \frac{(1 - q^{2\ell+2j})^{1/2}(1 - q^{2\ell+2i})^{1/2}}{(1 - q^{4\ell})^{1/2}(1 - q^{4\ell+2})^{1/2}}\,\varepsilon_{\ell-1/2,i-1/2,j-1/2}, \end{aligned} \tag{20}$$

$$\begin{aligned} \beta\,\varepsilon_{\ell,i,j} = {} & -q^{\ell+j}\,\frac{(1 - q^{2\ell-2j+2})^{1/2}(1 - q^{2\ell+2i+2})^{1/2}}{(1 - q^{4\ell+2})^{1/2}(1 - q^{4\ell+4})^{1/2}}\,\varepsilon_{\ell+1/2,i+1/2,j-1/2}\\ & + q^{\ell+i}\,\frac{(1 - q^{2\ell+2j})^{1/2}(1 - q^{2\ell-2i})^{1/2}}{(1 - q^{4\ell})^{1/2}(1 - q^{4\ell+2})^{1/2}}\,\varepsilon_{\ell-1/2,i+1/2,j-1/2}. \end{aligned} \tag{21}$$

This representation satisfies *par excellence* the requirement (ii) of Definition 2 with the full symmetry $U_q(su(2)) \otimes U_q(su(2))$.

We pass now to the question of spectral triples on $SU_q(2)$.

In [1] the Hilbert space of Dirac spinors is introduced as $\mathcal{H} = \mathbb{C}^2 \otimes L^2(SU_q(2))$ with the orthonormal basis chosen as

$$v^+_{\ell,ij} := C^{1/2,\ell,\ell+1/2}_{1/2,j-1/2,j} e_+ \otimes \varepsilon_{\ell,i,j-1/2} + C^{1/2,\ell,\ell+1/2}_{-1/2,j+1/2,j} e_- \otimes \varepsilon_{\ell,i,j+1/2},$$

$$v^-_{\ell,ij} := C^{1/2,\ell,\ell-1/2}_{1/2,j-1/2,j} e_+ \otimes \varepsilon_{\ell,i,j-1/2} + C^{1/2,\ell,\ell-1/2}_{-1/2,j+1/2,j} e_- \otimes \varepsilon_{\ell,i,j+1/2},$$

where $\ell \in \frac{1}{2}\mathbb{N}$, $i, j = -\ell, -\ell+1, \ldots, \ell; \ell = 0$, C denote the q-Clebsch–Gordan coefficients [1] and e_+, e_- is the standard orthonormal basis in $\mathbb{C}^2$ ($e_\pm = \epsilon_{1/2,\pm 1/2}$ if identifying $\mathbb{C}^2$ with $V_{1/2}$). The unitary representation of $\mathcal{A}$ acts on $\mathcal{H}$ as a tensor product of $\mathbb{I}_2$ and the left regular representation (20) and (21). A candidate for the Dirac operator is defined in [1] by declaring $v^+_{\ell,i,j}, v^-_{\ell,i,j}$ to be its eigenvectors with eigenvalues $[\ell]_{q^2}$ and $-[\ell+1]_{q^2}$, respectively. Unfortunately, it has unbounded commutators with the algebra A.

For that reason in [19] a modified operator $\tilde{D}$ has been proposed following the suggestion in [13], see also [12]. It has the same eigenvectors $v^+_{\ell,i,j}, v^-_{\ell,i,j}$ but the corresponding eigenvalues $\ell + \frac{1}{2}$ and $-(\ell + \frac{1}{2})$ respectively, have a linear growth. It turns out that its absolute value $|\tilde{D}|$ (though not $\tilde{D}$ itself) does satisfy the requirement of bounded commutators with the algebra A. Thus, $(A, \mathcal{H}, |\tilde{D}|)$ is a spectral triple. Unfortunately it fails to capture topological information of the underlying noncommutative space being $|\tilde{D}|$ a *positive* operator. Indeed the sign of $|\tilde{D}|$ is trivial and the corresponding Fredholm module has trivial pairing with K-theory. Accordingly, the Poincaré duality axiom does not hold.

Another spectral triple on $SU_q(2)$ was presented in [6]. Therein, the Hilbert space $\mathcal{H}$ is just (one copy of) $L^2(SU_q(2))$ with orthonormal basis (19). The unitary representation is just (20) and (21). The Dirac operator (up to a numerical factor) is

$$D\,\varepsilon_{\ell,i,j} = (1 - 2\,\delta_{i,\ell})\,\ell\,\varepsilon_{\ell,i,j},$$

where δ is the Kronecker symbol. It has bounded commutators with A. Due to the term with δ the condition (iii) of definition 1 is satisfied only for one copy of $U_q(su(2))$, which is the 'broken' symmetry of this spectral triple. Though D is not a positive operator, the Poincare' duality does not hold [8]. Another feature is that there is no 'classical' limit since $[D, h]$ blows up for some $h \in A$ as $q \to 1$. Moreover, J and the first order condition for D are not discussed.

Despite these peculiarities the spectral triple of [6] has been intensively studied in [11] from the analytical point of view. Therein, among other things, certain smooth subalgebra of $\mathcal{A}$ was defined, stable under holomorphic functional calculus. The regularity condition was verified and the pseudodifferential calculus constructed. Its algebra of complete symbols was determined (by computing the quotient by smoothing operators) and the cosphere bundle of $SU_q(2)$ constructed together with the geodesic flow. The summability condition was verified and the dimension spectrum computed to be $\Sigma = \{1, 2, 3\}$. The res_0 of the zeta function was explicitly computed and the explicit formula for the local index cocycle obtained.

Yet another spectral triple on $SU_q(2)$ appeared in [7]. Therein, the Hilbert space $\mathcal{H}$ has orthonormal basis ε_{ij} with $i \in \mathbb{N}$, $j \in \mathbb{Z}$. The (faithful) unitary representation is

$$\alpha\,\varepsilon_{i,j} = (1 - q^{2i})^{1/2}\,\varepsilon_{i-1,j}, \tag{22}$$

$$\beta\,\varepsilon_{i,j} = q^i\,\varepsilon_{i,j-1}. \tag{23}$$

It is equivariant under the action $\alpha \mapsto z\alpha$, $\beta \mapsto w\beta$ of the group $U(1) \times U(1)$, implemented on $\mathcal{H}$ by $e_{i,j} \mapsto z^i w^j\, e_{i,j}$. A class of $U(1) \times U(1)$-invariant Dirac operators was identified and among them

$$D\,\varepsilon_{i,j} = (i\,\mathrm{sign}(j) + j)\,\varepsilon_{i,j} \tag{24}$$

which is two-summable and not positive. In the 'classical' limit α degenerates to 0 and β becomes the simple shift. Moreover J and the first order condition for D are not established. However, interestingly the corresponding Connes–de Rham and the square integrable differential complexes have been computed.

Very recently a spectral triple appeared [18] with several nice properties.

The Hilbert space is $L^2(SU_q(2)) \otimes \mathbb{C}^2$ (note the order !). The representation of $U_q(su(2)) \otimes U_q(su(2))$ comes by coupling the first factor of $V = \bigoplus_{2l=0}^{\infty} V_l \otimes V_l$ with $V_{1/2}$. So in particular, $u \otimes u'$ is represented as

$$\bigoplus_{2\ell \geq 0} \sigma_\ell(u_1) \otimes \sigma_\ell(u') \otimes \sigma_{1/2}(u_2).$$

Next, this coupling is diagonalized by decomposing the tensor product with $\mathbb{C}^2$ as

$$\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow = \bigoplus_{2j \geq 0} W_j^\uparrow \oplus \bigoplus_{2j \geq 1} W_j^\downarrow.$$

(For consistency with the labeling of the representations by *spin* the index ℓ is offset by $\pm\frac{1}{2}$.) Hence, the orthonormal basis is

$$\epsilon_{j\mu n\downarrow} := C_{j\mu}\, \epsilon_{j^-\mu^+ n} \otimes \epsilon_{1/2,-1/2} + S_{j\mu}\, \epsilon_{j^-\mu^- n} \otimes \epsilon_{1/2,+1/2} \tag{25}$$

for the labels $j = l + \frac{1}{2}$, $\mu = m - \frac{1}{2}$, with $\mu = -j, \ldots, j$ and $n = -j^-, \ldots, j^-$ and

$$\epsilon_{j\mu n\uparrow} := -S_{j+1,\mu}\, \epsilon_{j^+\mu^+ n} \otimes \epsilon_{1/2,-1/2} + C_{j+1,\mu}\, \epsilon_{j^+\mu^- n} \otimes \epsilon_{1/2,+1/2} \tag{26}$$

for the labels $j = l - \frac{1}{2}$, $\mu = m - \frac{1}{2}$, with $\mu = -j, \ldots, j$ and $n = -j^+, \ldots, j^+$. Here the coefficients are

$$C_{j\mu} := q^{-(j+\mu)/2}\,[j-\mu]^{1/2}[2j]^{-1/2}, \qquad S_{j\mu} := q^{(j-\mu)/2}\,[j+\mu]^{1/2}[2j]^{-1/2} \tag{27}$$

and shorthand notation $k^\pm := k \pm \frac{1}{2}$ is adopted. (Notice that there are no $\downarrow$ vectors for $j = 0$.)

In order to simplify the formulae for the representation of A on $\mathcal{H}$, the pair of spinors is juxtaposed as

$$|j\mu n\rangle\rangle := \begin{pmatrix} \epsilon_{j\mu n\uparrow} \\ \epsilon_{j\mu n\downarrow} \end{pmatrix} \tag{28}$$

for $j \in 1/2\mathbb{N}$, with $\mu = -j, \ldots, j$ and $n = -j - \frac{1}{2}, \ldots, j + \frac{1}{2}$, with the convention that the lower component is zero when $n = \pm\left(j + \frac{1}{2}\right)$ or $j = 0$. Furthermore, a matrix with scalar entries

$$\tau = \begin{pmatrix} \tau_{\uparrow\uparrow} & \tau_{\uparrow\downarrow} \\ \tau_{\downarrow\uparrow} & \tau_{\downarrow\downarrow} \end{pmatrix},$$

is understood to act on $|j\mu n\rangle\rangle$ by the rule:

$$
\begin{aligned}
\tau\epsilon_{j\mu n\uparrow} &= \tau_{\uparrow\uparrow}\epsilon_{j\mu n\uparrow} + \tau_{\downarrow\uparrow}\epsilon_{j\mu n\downarrow},\\
\tau\epsilon_{j\mu n\downarrow} &= \tau_{\downarrow\downarrow}\epsilon_{j\mu n\downarrow} + \tau_{\uparrow\downarrow}\epsilon_{j\mu n\uparrow}.
\end{aligned}
\tag{29}
$$

The unitary representation π of A on $L^2(SU_q(2)) \otimes \mathbb{C}^2$ is the tensor product of the left regular representation (20) and (21) and $\mathbb{I}_2$. On the basis (28) it reads explicitly

$$
\begin{aligned}
\alpha\,|j\mu n\rangle\rangle &= \alpha^+_{j\mu n}|j^+\mu^+n^+\rangle\rangle + \alpha^-_{j\mu n}|j^-\mu^+n^+\rangle\rangle,\\
\beta\,|j\mu n\rangle\rangle &= \beta^+_{j\mu n}|j^+\mu^+n^-\rangle\rangle + \beta^-_{j\mu n}|j^-\mu^+n^-\rangle\rangle,\\
\alpha^*\,|j\mu n\rangle\rangle &= \tilde{\alpha}^+_{j\mu n}|j^+\mu^-n^-\rangle\rangle + \tilde{\alpha}^-_{j\mu n}|j^-\mu^-n^-\rangle\rangle,\\
\beta^*\,|j\mu n\rangle\rangle &= \tilde{\beta}^+_{j\mu n}|j^+\mu^-n^+\rangle\rangle + \tilde{\beta}^-_{j\mu n}|j^-\mu^-n^+\rangle\rangle,
\end{aligned}
\tag{30}
$$

where $\alpha^\pm_{j\mu n}$ and $\beta^\pm_{j\mu n}$ are, up to phase factors depending only on j, the following triangular 2×2 matrices:

$$
\begin{aligned}
\alpha^+_{j\mu n} &= q^{(1/2-\mu-n)/2}\,[j+\mu+1]^{1/2}\\
&\quad\times\begin{pmatrix} q^{j+1/2}\,\dfrac{[j+n+3/2]^{1/2}}{[2j+2]} & 0\\ q^{-1/2}\,\dfrac{[j-n+1/2]^{1/2}}{[2j+1]\,[2j+2]} & q^{j}\,\dfrac{[j+n+1/2]^{1/2}}{[2j+1]}\end{pmatrix},\\
\alpha^-_{j\mu n} &= q^{(1/2-\mu-n)/2}\,[j-\mu]^{1/2}\\
&\quad\times\begin{pmatrix} q^{-j-1}\,\dfrac{[j-n+1/2]^{1/2}}{[2j+1]} & -q^{-1/2}\,\dfrac{[j+n+1/2]^{1/2}}{[2j]\,[2j+1]}\\ 0 & q^{-j-1/2}\,\dfrac{[j-n-1/2]^{1/2}}{[2j]}\end{pmatrix},\\
\beta^+_{j\mu n} &= q^{(-\mu-n-1/2)/2}\,[j+\mu+1]^{1/2}\\
&\quad\times\begin{pmatrix} \dfrac{[j-n+3/2]^{1/2}}{[2j+2]} & 0\\ -q^{j+1}\,\dfrac{[j+n+1/2]^{1/2}}{[2j+1]\,[2j+2]} & q^{1/2}\,\dfrac{[j-n+1/2]^{1/2}}{[2j+1]}\end{pmatrix},\\
\beta^-_{j\mu n} &= q^{(-\mu-n-1/2)/2}\,[j-\mu]^{1/2}\\
&\quad\times\begin{pmatrix} -q^{1/2}\,\dfrac{[j+n+1/2]^{1/2}}{[2j+1]} & -q^{-j}\,\dfrac{[j-n+1/2]^{1/2}}{[2j]\,[2j+1]}\\ 0 & -\dfrac{[j+n-1/2]^{1/2}}{[2j]}\end{pmatrix},
\end{aligned}
\tag{31}
$$

and the remaining matrices are the hermitian conjugates

$$\tilde{\alpha}^{\pm}_{j\mu n} = (\alpha^{\mp}_{j^{\pm}\mu^{-}n^{-}})^{\dagger}, \qquad \tilde{\beta}^{\pm}_{j\mu n} = (\beta^{\mp}_{j^{\pm}\mu^{-}n^{+}})^{\dagger}.$$

This (spinorial) representation is equivariant (cf. (ii) of Definition 2) under the full symmetry $U_q(su(2)) \otimes U_q(su(2))$.

(Note that the decomposition of $\mathbb{C}^2 \otimes V$ as taken in [1,19] differs from $V \otimes \mathbb{C}^2$ by the q-Clebsch–Gordan coefficients according to the rule

$$C_q \begin{pmatrix} j & l & m \\ r & s & t \end{pmatrix} = C_q \begin{pmatrix} l & j & m \\ -s & -r & -t \end{pmatrix}$$

which amounts to a substitution of q by q^{-1} in the coefficients C and S (27). However, $\mathbb{C}^2 \otimes V$ is not equivariant in the sense of Definition 2.)

As far as the Dirac operator is concerned, it is diagonal with linear eigenvalues:

$$D\epsilon_{j\mu n\uparrow} = d_j^{\uparrow}\, \epsilon_{j\mu n\uparrow}, \qquad D\epsilon_{j\mu n\downarrow} = d_j^{\downarrow}\, \epsilon_{j\mu n\downarrow}, \tag{32}$$

where

$$d_j^{\uparrow} = c_1^{\uparrow} j + c_2^{\uparrow}, \qquad d_j^{\downarrow} = c_1^{\downarrow} j + c_2^{\downarrow}, \tag{33}$$

with $c_1^{\uparrow}, c_2^{\uparrow}, c_1^{\downarrow}, c_2^{\downarrow}$ independent of j. (The multiplicities are $(2j+1)(2j+2)$ and $2j(2j+1)$.)

Such D is self-adjoint on a natural domain and has a compact resolvent. The linearity of eigenvalues suffices to check that $[D, x]$, $x \in A$, are bounded. Moreover D is invariant under the full $U_q(su(2)) \otimes U_q(su(2))$ by construction.

In addition, if $c_1^{\downarrow} = -c_1^{\uparrow}$, $c_2^{\downarrow} = -c_2^{\uparrow} + c_1^{\uparrow}$ then the spectrum of D coincides with that of the classical Dirac operator $\not{D}$ on the round sphere S^3, up to rescaling and addition of a constant. Thus, this spectral triple is an isospectral deformation of $(C^{\infty}(S^3), \mathcal{H}, \not{D})$, and in particular, its spectral dimension is 3.

Altogether, $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ forms a 3^+-summable spectral triple, equivariant under the full symmetry $U_q(su(2)) \otimes U_q(su(2))$, which is the whole rationale of its construction.

There is also an antilinear conjugation operator J on $\mathcal{H}$ which is defined explicitly on the orthonormal spinor basis by

$$\begin{aligned} J\,\epsilon_{j\mu n\uparrow} &:= i^{2(2j+\mu+n)}\,\epsilon_{j,-\mu,-n,\uparrow}, \\ J\,\epsilon_{j\mu n\downarrow} &:= i^{2(2j-\mu-n)}\,\epsilon_{j,-\mu,-n,\downarrow}. \end{aligned} \tag{34}$$

It is immediate that J is antiunitary and that $J^2 = -1$, since each $4j \pm 2(\mu + n)$ is an odd integer. Moreover, $JDJ^{-1} = D$ since D and J diagonalize on their common eigenspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$.

Besides these usual properties J departs from the axiom scheme for real spectral triples. Namely it does not map $\pi'(A)$ to its commutant, but it does (!) modulo the two-sided ideal $\mathcal{G}$ of $\mathcal{B}(\mathcal{H})$ generated by the (positive trace-class) operator $L_q|j\mu n\rangle\rangle := q^j\,|j\mu n\rangle\rangle$. ($\mathcal{G} \subset \mathcal{K}$ is contained in all ideals of infinitesimals of order α for any $\alpha > 0$).

More precisely, for any $x, y \in A$,

$$[JxJ^{-1}, y] \in \mathcal{G}. \tag{35}$$

Also the first order condition holds only up to $\mathcal{G}$, that is for all $x, y \in A$,

$$[JxJ^{-1}, [D, y]] \in \mathcal{G}. \tag{36}$$

We remark that in a sense this spectral triple realizes the suggestion in [13] (see also [12]). However differently to [19] (and [1]), the spinor representation is constructed by tensoring $L^2(SU_q(2))$ by $\mathbb{C}^2$ on the right rather than on the left. Though the Clebsch–Gordan decomposition looks similar, the different asymptotics of the appropriate Clebsch–Gordan coefficients leads to bounded commutator $[D, \pi(x)]$ instead of unbounded one for some x in [19] (there the off-diagonal terms of $\pi(x)$ are not compact but rather estimated by $\epsilon > 0$ from below). The origin of such a drastic difference is the (unbounded) non-co-commutativity of $U_q(su(2))$.

We refer to [29] for the analysis of this example along the lines of [11] culminating with the local index formulae.

4. Quantum two-spheres

We pass now to two-dimensional examples.

4.1. *The standard quantum sphere*

The algebra A of the standard Podleś quantum sphere [25] can be defined in terms of generators $a = a^*, b, b^*$ and relations $(0 \le q < 1)$

$$ab = q^2 ba, \quad ab^* = q^{-2} b^* a, \quad bb^* = q^{-2} a(1-a), \quad b^* b = a(1 - q^2 a). \tag{37}$$

The associated C^*-algebra is isomorphic to the minimal unitization of the compacts $\mathcal{K}$. It has one classical point given by the character $a \mapsto 0, b \mapsto 0$ and K-groups are $K_0 = \mathbb{Z}^2$, $K_1 = 0$.

The algebra A was discovered as a $A(SU_q(2))$-comodule algebra, but it can be also viewed as a subalgebra of $A(SU_q(2))$ generated by

$$b = -\alpha^* \beta, \quad b^* = -\beta^* \alpha, \quad a = \beta\beta^* = a^*. \tag{38}$$

The standard Podleś quantum sphere is not only a homogeneous $SU_q(2)$-space, but also a quotient space $SU_q(2)/U(1)$. (In fact the quantum Hopf fibration and monopole connection are well known.)

The $\triangleright$ action (16) of $U_q(su(2))$ descends to A, up to equivalence it reads

$$\begin{aligned}
&e \triangleright b = -(q^{1/2} + q^{-3/2})a + q^{-3/2}, \quad e \triangleright b^* = 0, \quad e \triangleright a = q^{-1/2} b^*,\\
&k \triangleright b = qb, \quad k \triangleright b^* = q^{-1} b^*, \quad k \triangleright a = a,\\
&f \triangleright b = 0, f \triangleright b^* = (q^{3/2} + q^{-1/2})a - q^{-1/2}, \quad f \triangleright a = -q^{1/2} b.
\end{aligned} \tag{39}$$

However this is not the case for $\triangleleft$. Indeed setting for $m \in \frac{1}{2}\mathbb{Z}$

$$A_m = \{h \in A(SU_q(2)) | h \triangleleft k = q^m h\}$$

(in particular $A_0 = A$) it is easy to see that $A_m \triangleleft e = A_{m+1}$ and $A_m \triangleleft f = A_{m-1}$. (Nevertheless $\triangleleft$ will still play a rôle in the sequel.)

A spectral triple on the standard Podleś sphere has been constructed in [16]. The Hilbert space $\mathcal{H}$ has the orthonormal basis $\epsilon_{\ell,m,s}$ with $\ell \in \mathbb{N} + 1/2$, $m \in \{-\ell, -\ell+1, \ldots, \ell\}$ and $s \in \{-1, 1\}$. The bounded representation π of A on $\mathcal{H}$ reads:

$$\begin{aligned} a\epsilon_{\ell,m,s} &= a^{+}_{\ell,m,s}\epsilon_{\ell+1,m,s} + a^{0}_{\ell,m,s}\epsilon_{\ell,m,s} + a^{-}_{\ell,m,s}\epsilon_{\ell-1,m,s}, \\ b\epsilon_{\ell,m,s} &= b^{+}_{\ell,m,s}\epsilon_{\ell+1,m+1,s} + b^{0}_{\ell,m,s}\epsilon_{\ell,m+1,s} + b^{-}_{\ell,m,s}\epsilon_{\ell-1,m+1,s}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} a^{+}_{\ell,m,s} =& -q^{2\ell+m+1-s/2} \\ & \times \frac{(1-q^{2\ell-2m+2})^{1/2}(1-q^{2\ell+2m+2})^{1/2}}{(1-q^{4\ell+4})^{1/2}(1+q^{2\ell+3}+q^{2\ell+1}-q^{6\ell+7}-q^{6\ell+5}-q^{8\ell+8})^{1/2}}, \\ a^{0}_{\ell,m,s} =& (1+q^2)^{-1} \\ & + \frac{(1-q^{4\ell+2}+q^{2\ell+2m}+q^{2\ell+2m+2})((1-q^{2\ell-1})\times(1-q^{2\ell+3})-sq^{2\ell+s}(1-q^2))}{(1+q^2)(1-q^{4\ell})(1-q^{4\ell+4})}, \\ a^{-}_{\ell,m,s} =& -q^{2\ell+m-1-s/2}\frac{(1-q^{2\ell-2m})^{1/2}(1-q^{2\ell+2m})^{1/2}}{(1-q^{4\ell})^{1/2}(1+q^{2\ell+1}+q^{2\ell-1}-q^{6\ell+1}-q^{6\ell-1}-q^{8\ell})^{1/2}}, \end{aligned} \tag{41}$$

$$\begin{aligned} b^{+}_{\ell,m,s} &= q^{\ell-s/2}\frac{(1-q^{2\ell+2m+4})^{1/2}(1-q^{2\ell+2m+2})^{1/2}}{(1-q^{4\ell+4})^{1/2}(1+q^{2\ell+3}+q^{2\ell+1}-q^{6\ell+7}-q^{6\ell+5}-q^{8\ell+8})^{1/2}}, \\ b^{0}_{\ell,m,s} &= q^{m+\ell}\frac{\begin{array}{c}(1-q^{2\ell+2m+2})^{1/2}(1-q^{2\ell-2m})^{1/2}(sq^{2\ell+s}(1-q^2) \\ -(1-q^{2\ell-1})(1-q^{2\ell+3}))\end{array}}{(1-q^{4\ell})(1-q^{4\ell+4})}, \\ b^{-}_{\ell,m,s} &= -q^{2m+3\ell-s/2}\frac{(1-q^{2\ell-2m})^{1/2}(1-q^{2\ell-2m-2})^{1/2}}{(1-q^{4\ell})^{1/2}(1+q^{2\ell+1}+q^{2\ell-1}-q^{6\ell+1}-q^{6\ell-1}-q^{8\ell})^{1/2}}. \end{aligned} \tag{42}$$

(The formulae for b^* are similar.)

The action of $U_q(su(2))$ on A is as in (39). The $*$-representation of $U_q(su(2))$ on $\mathcal{H}$ is a direct sum of the half integer spin ($\ell \in \mathbb{N} + 1/2$) finite dimensional irreducible representations σ_ℓ (18) of $U_q(su(2))$ with multiplicity 2 ($s = \pm 1$). This corresponds to the simplest possible even spectral triple; higher multiplicities can be dealt accordingly.

With this, π is a direct sum of (the only) two inequivalent $U_q(su(2))$-equivariant representations $\pi_\pm$ of $A(S_q^2)$ on $\mathcal{H}_\pm$ ($s = \pm 1$). In fact, $\pi_\pm$ are irreducible representations of $A(S_q^2) \rtimes U_q(su(2))$.

Next, in [16] the following operators are constructed: the grading

$$\gamma \epsilon_{\ell,m,s} = s\, \epsilon_{\ell,m,s},$$

the reality structure

$$J \epsilon_{\ell,m,s} = i^{2m} \epsilon_{\ell,-m,-s} \tag{43}$$

(antilinear isometry satisfying $J^2 = -1$, $\gamma J = -J\gamma$ and sending $A(S_q^2)$ to its commutant), and the Dirac operator

$$D \epsilon_{\ell,m,s} = \left[\ell + \tfrac{1}{2}\right] \epsilon_{\ell,m,-s}, \tag{44}$$

satisfying $D\gamma = -\gamma D$ and $DJ = JD$ and $Du = uD$, $\forall u \in U_q(su(2))$.

In [16] it is shown that such data $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ define a (compact) real even $U_q(su(2))$-equivariant spectral triple, which satisfies the first order condition. Moreover, with the action (39) and the representation (40)–(42), D is unique up to multiplication of the r.h.s. of (44) respectively by z (or $\bar{z}$) when $s = +1$ (or $s = -1$), where $z \in \mathbb{C} \setminus \{0\}$.

Evidently D is a (unbounded) self-adjoint operator defined on a natural dense domain in $\mathcal{H}$ consisting of $\psi = \sum_{\ell,m,s} c_{\ell,m,s} \epsilon_{\ell,m,s}$, $c_{\ell,m,s} \in \mathbb{C}$, such that $\sum_{\ell,m,s} (1 + [\ell + \frac{1}{2}]^2) |c_{\ell,m,s}|^2 < \infty$.

The limit $q \to 1$ is in agreement with the classical spectral triple on S^2, identifying $\epsilon_{\ell,m,\pm}$ with $Y^{\pm}_{\ell,m}$. Also, J coincides with the charge conjugation on spinors and D has the correct $q = 1$ limit.

The geometrical meaning of $\mathcal{H}$, J, D given above is quite clear. Recall that classically Dirac spinor ψ on S^2 is a section of certain rank two vector bundle associated to the bundle of *spin frames*, which is nothing but the Hopf bundle $U(1) \to SU(2) \to S^2$. Equivalently, ψ can be thought of as $U(1)$-equivariant (under $z \to z^{-1} \oplus z$) $\mathbb{C}^2$-valued function on $SU(2)$, or equivalently, as $-\frac{1}{2} \oplus \frac{1}{2}$ eigenvector of $\triangleleft\, H$ (Cartan generator of $su(2)$). Of course the space $\Gamma = \{\psi\}$ of such polynomial spinors is a finite projective module over A.

The q-deformation of these bundles (and more) is well known [2,22] and we can view $SU_q(2)$ as the bundle of spin q-frames, and Dirac spinors Γ_q as $q^{-1/2} \oplus q^{1/2}$ eigenspace of the action $\triangleleft$ of K on $A(SU_q(2))$. Explicitly,

$$\Gamma_q = A_{-1/2} \oplus A_{1/2} = \mathrm{span}_A\{\alpha, \beta\} \oplus \mathrm{span}_A\{\alpha^*, \beta^*\}.$$

It is a finite projective module over A, with a projector in $\mathrm{Mat}(2, A) \oplus \mathrm{Mat}(2, A)$ being

$$\begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix} \oplus \begin{pmatrix} \alpha^*\alpha & q\alpha^*\beta \\ q\beta^*\alpha & q^2\beta^*\beta \end{pmatrix} \tag{45}$$

$$= \begin{pmatrix} 1-q^2a & -qb^* \\ -qb & a \end{pmatrix} \oplus \begin{pmatrix} 1-a & -qb \\ -qb^* & q^2a \end{pmatrix}. \tag{46}$$

The Hilbert scalar product comes by restriction from $L^2(SU_q(2))$ and sending $\epsilon_{\ell,m,s} \to [2\ell+1]^{-1/2}q^{-s/2}\varepsilon_{\ell,s/2,m}$ gives unitary isomorphism. This explains the relation of $\mathcal{H}_\pm$ and $\mathcal{H}$ to q-deformed Hopf bundles.

With this setup the above Dirac operator is just $\left(\begin{smallmatrix} 0 & \triangleleft\, e \\ \triangleleft\, f & 0 \end{smallmatrix}\right)$. Then the invariance (5) is clear and since $\triangleleft\, e$ and $\triangleleft\, f$ are twisted derivations it is immediate to see [28,24] that for $x \in A$

$$[D,x] = \begin{pmatrix} 0 & q^{1/2}h \triangleleft e \\ q^{-1/2}h \triangleleft f & 0 \end{pmatrix} \tag{47}$$

are bounded.

We comment on other axioms. Concerning the dimension axiom the eigenvalues of $|D|$ are $\pm[k]_q$ with degeneracy $4k$, where $k := \ell + 1/2 \in \mathbb{N}$. Thus the eigenvalues of $|D|^{-z}$ decrease exponentially as $k \to \infty$ for $Re z > 0$ (recall that $q < 1$). By summing the derivative of the geometric series finite-summability holds, actually ϵ-summability for any $\epsilon > 0$

$$\mathrm{Trace}_{\mathcal{H}}|D|^{-\epsilon} < \frac{4(q-q^{-1})^\epsilon q^\epsilon}{(q^\epsilon - 1)^2}.$$

Next $\sigma_N/\log N \to 0$ for all $z > 0$, $\to \infty$ for $z \le 0$. So, loosely speaking, the metric dimension of this geometry is '0_+', which matches the known drop by two of the cohomological dimension of q-spaces. More precisely, the dimension spectrum Σ should be studied, e.g. the singularities of $\zeta_\beta(z) = \mathrm{Trace}_{\mathcal{H}}(\beta\,|D|^{-z})$. For $\beta = 1$ its most divergent part goes as

$$\sim 4(q-q^{-1})^z \sum_{k\in\mathbb{N}} kq^{zk} \sim \frac{4(q-q^{-1})^{2\pi in/h}}{(z-2\pi in/h)^2}$$

as $z \sim 2\pi in/h$, where $h = \log q$, $\forall n \in \mathbb{N}$ (at least for $Re(z) \ge 0$). This suggests that Σ should contain a (discrete) infinite lattice $2\pi in/h$, $n \in \mathbb{Z}$ of *double* poles on the imaginary axis.

That simple poles are excluded could be expected from the divergence of $\sigma_N/\log N$ at $z = 0$, due to Tauberian theorem. Recall also that complex poles may appear on fractals and multiple poles on singular manifolds.

Concerning the reality requirement J is tailored as in the (naive) dimension $d = 2$ (as indicated by the signs in $J^2 = -1$, $JD = +DJ$ and $J\gamma = -\gamma J$), which has at least the same parity as metric dimension.

The regularity axiom does not hold [24] in its rigorous form but it looks as if there might be some substitute for it since for $h \in A_m$, $|D|^z h |D|^{-z} - q^{mz}h$ is of order -1. Thus, by (47) and recalling that $h \triangleleft e \in A_1$ and $h \triangleleft f \in A_{-1}$ for $h \in A_0 = A$, it appears that the principal symbol of $|D|$ is not a scalar. But one can still work with this sort of 'spin anomaly' by using q-mutators since

$$|D|^{-z}[D,h] - \begin{pmatrix} q^z & 0 \\ 0 & q^{-z} \end{pmatrix}[D,h]|D|^{-z} = \kappa(z)|D|^{-z-1},$$

where $\kappa(z)$ are bounded operators analytic in z. In fact a 'twisted local index formula' for $SU_q(2)$ was worked out in [24] in the framework of twisted cyclic cohomology.

Concerning the orientation axiom perhaps some modification of [28] containing some proofs due to Heckenberger, and of [24] may be helpful. They show, that $d\alpha := i[D, \pi(\alpha)]$ gives the unique two-dimensional covariant first order differential calculus of [26].

The related universal differential calculus contains a A-central two-form, with which one can associate (via the Haar measure χ, such that $\chi(ab) = \chi(b\sigma(a))$, where σ is the algebra automorphism of A given by $\sigma = K^2 \triangleright$) a nontrivial σ-*twisted* cyclic Hochschild two-cocycle τ on A

$$\tau(\alpha,\beta,\gamma) := \chi(\alpha(\beta \triangleleft e)(\gamma \triangleleft f) - q^2\alpha(\beta \triangleleft f)(\gamma \triangleleft e))$$

$$= (q-q^{-1})^{-1}\log q\, \mathrm{Res}_{z=2}\, \mathrm{Tr}_{\mathcal{H}} \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix} K^2|D|^{-z}\alpha[D,\beta][D,\gamma]$$

[28] (see also Versions 1 and 2 of [19]).

4.2. *The equatorial quantum sphere*

The algebra of equatorial Podleś sphere can be written as

$$ba = q^2ab, \quad b^*b + a^2 = 1, \quad bb^* + q^4a^2 = 1.$$

The classical subset is (the 'equator') S^1 given by the characters $a \mapsto 0$, $b \mapsto \lambda$ with $|\lambda| = 1$. The related C^*-algebra is extension of $C(S^1)$ by $\mathcal{K} \oplus \mathcal{K}$ and $K_0 = \mathbb{Z}^2$, $K_1 = 0$.

The symmetry used for the equivariance is now

$$k \triangleright a = a,\ e \triangleright a = q^{-1/2}b^*,\ f \triangleright a = -q^{-3/2}b,$$
$$k \triangleright b = q^{-1}b,\ e \triangleright b = -(1+q^{-2})q^{5/2}a,\ f \triangleright b = 0,$$
$$k \triangleright b^* = qb^*,\ e \triangleright b^* = 0,\ f \triangleright b^* = (1+q^{-2})q^{3/2}a.$$

In [17] the data $\mathcal{H}$, γ and J are as in Section 4.1 and the representation (up to equivalence) is $\pi = \pi_+ \oplus \pi_-$ where $\pi_\pm$ are the (only) two irreducible $U_q(su(2))$-equivariant $*$-representations

$$a\,\epsilon_{\ell,m} = -q^{\ell-m}\frac{(1-q^{2\ell-2m+2})^{1/2}(1-q^{2\ell+2m+2})^{1/2}}{1-q^{4\ell+4}}\,\epsilon_{\ell+1,m}$$
$$\mp q^{2\ell-1}\frac{(1-q^2)(1+q^{4\ell+2}-q^{2\ell-2m}-q^{2\ell-2m+2})}{(1-q^{4l})(1-q^{4l+4})}\epsilon_{\ell,m}$$
$$-q^{\ell-m-1}\frac{(1-q^{2\ell-2m})^{1/2}(1-q^{2\ell+2m})^{1/2}}{1-q^{4\ell}}\,\epsilon_{\ell-1,m}, \tag{48}$$

$$
\begin{aligned}
b\,\epsilon_{\ell,m} &= q^{2\ell-2m+1}\frac{(1-q^{2\ell+2m+2})^{1/2}(1-q^{2\ell+2m+4})^{1/2}}{1-q^{4\ell+4}}\epsilon_{\ell+1,m+1}\\
&\pm q^{3\ell-m-1}\frac{(1-q^4)(1-q^{2\ell+2m+2})^{1/2}(1-q^{2\ell-2m})^{1/2}}{(1-q^{4l})(1-q^{4l+4})}\epsilon_{\ell,m+1}\\
&-\frac{(1-q^{2\ell-2m})^{1/2}(1-q^{2\ell-2m-2})^{1/2}}{1-q^{4\ell}}\,\epsilon_{\ell-1,m+1}.
\end{aligned}
\tag{49}
$$

They are equivalent to the representation obtained by restricting the representation (20) and (21) of $A(SU_q(2))$ to the subalgebra A and by restricting $\mathcal{H}$ to the (L^2-completion) of certain vector spaces (left A-modules) constructed in [3].

It is shown in [17] that J does not map $\pi(A)$ to its commutant, but it does modulo the ideal $\mathcal{G} \subset \mathcal{K}$ generated by the operator q^l on $\epsilon_{\ell,m,s}$. More precisely, for any $x, y \in A$,

$$[JxJ^{-1}, y] \in \mathcal{G}. \tag{50}$$

The Dirac operator is

$$D\epsilon_{\ell,m,s} = \left(\ell + \tfrac{1}{2}\right)\epsilon_{\ell,m,-s}. \tag{51}$$

It is unique (up to rescaling and addition of a constant) under the same postulates as in Section 4.1 except the first order condition is required only up to $\mathcal{G}$, that is for all $x, y \in A$,

$$[JxJ^{-1}, [D, y]] \in \mathcal{G}. \tag{52}$$

More importantly, for any $x \in A$, the commutators $[D, x]$ are bounded. Moreover, it is evident that D is self-adjoint on a natural domain in $\mathcal{H}$ and has compact resolvent. Since the eigenvalues of $|D|$ are $k = \ell + \frac{1}{2} \in \mathbb{N}$ with multiplicity $4k$ the deformation is isospectral and the dimension requirement is satisfied with the spectral dimension of $(A, \mathcal{H}, D)$ being $n = 2$.

4.3. *The generic quantum sphere*

The algebra of generic Podleś sphere $S^2_{q,c}$ can be written as

$$ba = q^2ab, \quad b^*b + a^2 - a = c1, \quad bb^* + q^4a^2 - q^2a = c1,$$

where $q < 1$, $0 < c < \infty$. The classical subset is S^1 given by the characters $a \mapsto 0$, $b \mapsto \lambda$ with $|\lambda| = c$. The related C^*-algebra is extension of $C(S^1)$ by $\mathcal{K} \oplus \mathcal{K}$ and $K_0 = \mathbb{Z}^2$, $K_1 = 0$.

A spectral triple on $S^2_{q,c}$ appeared in [7]. Therein, the Hilbert space $\mathcal{H}$ has orthonormal basis $e_{n,s}$ with $n \in \mathbb{N}$, $s \in \{-1, +1\}$. The (faithful) unitary representation is

$$a\,\epsilon_{n,s} = \left(\tfrac{1}{2} + s(c + \tfrac{1}{4})^{1/2}\right) q^{2n}\,\epsilon_{n,s}, \tag{53}$$

$$b\,\epsilon_{n,s} = \left(\left(\tfrac{1}{2} + s(c + \tfrac{1}{4})^{1/2}\right) q^{2n} - \left(\tfrac{1}{2} + s(c + \tfrac{1}{4})^{1/2}\right)^2 q^{4n} + c\right)^{1/2}\epsilon_{n,s}. \tag{54}$$

It is equivariant under the action $\alpha \mapsto z\alpha$, $\beta \mapsto w\beta$ of the group $U(1) \times U(1)$, implemented on $\mathcal{H}$ by $e_{i,j} \mapsto z^i w^j e_{i,j}$. A class of $U(1) \times U(1)$-invariant Dirac operators was identified and among them,

$$D\,\epsilon_{n,s} = n\,\epsilon_{n,-s}, \tag{55}$$

which is one-summable, not positive and has bounded commutators with the algebra. This spectral triple is even, $\gamma\epsilon_{n,s} = s\,\epsilon_{n,s}$, has nontrivial Chern character but J and the first order condition are not considered. However, interestingly the corresponding Connes–de Rham differential complex has been computed.

5. Final comments

It is worth mentioning that D, or its spectrum, taken individually carries only part of the available geometric information. It is the interplay of all the data of the spectral triple (also J and γ) on the Hilbert space, that imposes some really stringent restrictions and produces the spectacular consequences.

The examples of some recently constructed spectral triples on quantum spheres we have described dissolve the widespread belief that Connes' approach to noncommutative geometry does not match quantum-group theory.

The last example in Section 3 acting on two copies of the L_2 space is odd, isospectral to the classical Dirac operator, three-summable and has the $U_q(sl(2)) \otimes U_q(sl(2))$ symmetry. It exhibits a variation of 'up to infinitesimals' of the reality condition and the first order condition, which was anticipated by the even, isospectral, two-summable, $U_q(sl(2))$-equivariant example in Section 4.2. The (even) example of Section 4.1 is not isospectral but it satisfies the reality and the first order axioms in the strict sense.

We believe the isospectral deformations will prove useful in the q-geometry and will be omnipresent on other q-deformed spaces. In fact this point is currently studied on general Podleś sphere (including the standard one).

There are several interesting open mathematical problems regarding the analytic properties of these spectral triples (regularity and finiteness) as well as the algebraic ones (orientation and Poincaré duality) and of the local index formulae on two-spheres.

From the more physical point of view the interesting questions regard e.g. the construction of a wider class of spectral triples and of the Yang–Mills and gravitation theories on q-deformed spaces. In particular the action functional (15) on spectral triples, the gravitational part of which exhibits a huge symmetry including q-deformations of A with isospectral deformations of D, and its extrema should be further studied.

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